Analysis 1A — Tutorial 5

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# Introduction

Here is the material to accompany the Analysis Tutorial in Week 5. Alternative formats can be downloaded by clicking the download icon at the top of the page. As usual, send comments and corrections to [Christian Jones (caj50)](mailto:caj50@bath.ac.uk). To return to the homepage, click [here](http://caj50.github.io/tutoring.html).

# Lecture Recap

## Sequences

### Two Useful Theorems

Last week, we were introduced to the idea of sequences and what it means for a sequence to *converge*. The definition of this is repeated below.

Definition 1.1 (Sequence Convergence)

A sequence converges to a real number as , written as either , or if

Using this definition, we can establish two theorems which will really help us when establishing further results about sequences.

Theorem 1.1 (Preservation of Non-Strict Inequalities)

Let and be sequences and let be such that and as . If , then .

There are two good uses for this theorem. The first says that non-negative sequences should have non-negative limits (which is something you might expect). Before we state the second, we mention one more thing, which is **not true**:

Non-theorem:

Let and be sequences and let be such that and as . If , then .

To see why this is false, consider the sequences defined by and . We note that each is strictly less than each corresponding , but we find that

The second reason why Theorem 1.1 is so important, is that it gives us this second theorem[[1]](#footnote-28):

Theorem 1.2 (Uniqueness of Limits)

If is convergent with and as , then .

### Bounded Sequences

Much like we have done with sets, we can formulate a definition which allows us to ‘trap’ sequences.

Definition 1.2 (Bounded Sequence)

A sequence is bounded if there exists such that .

If you prefer to think diagramatically, this says we can trap the sequence within a strip of width around . More importantly, this leads to the idea that *all convergent sequences are bounded*. Note that this is equivalent to saying that if a sequence is not bounded, then it is not convergent.

### Algebra of Limits

Using the definition to prove all limits would be an incredibly boring way to go through this course. Luckily, there are a few general results we can prove which make our lives so much easier. This is known as the *algebra of limits* (AoL).

Theorem 1.3 (Algebra of Limits)

Let and let and be sequences with and as . Then:

1. ,
2. ,
3. ,
4. If and , .

# Hints

As per usual, here’s where you’ll find the problem sheet hints!

* 1. At this stage of the course, you’ll have to use the definition of convergence. Apply this, and follow the hint on the sheet.
  2. This one you can apply AoL! However, you should explain why you can use AoL, namely: which convergence results from lectures are you applying?

1. Again, this is one you have to use the definition on (for the time being, at least). Remember, if you make the denominator of a fraction smaller, you’ll make the overall fraction bigger.
2. For the first bit, you can use AoL! For the second, you’ll have to use the definition, making a **specific** choice of
3. This is very similar to the first question from tutorials, so here’s a few (vague-ish) hints:
   1. Try and find expressions to simplify the sums.
   2. Again, you’ll want to find an explicit expression for the sum of square numbers.
   3. Firstly, what do you get if you factorise ? The result of Tutorial Question 2 will also be useful here.
   4. Evaluate
   5. Simplify the expression first.
   6. You’ll have to use the definition again here.
4. This is another definition question. Use the knowledge that converges (i.e. what does this mean explicitly?) to conclude something about the convergence of .

1. Feel free to ignore this footnote, but there are areas of maths where limits are not unique. This is usually in the realm of topology, which you can take in Year 3 [(MA30055)](https://www.bath.ac.uk/catalogues/2023-2024/ma/MA30055.html). Luckily for us, everything behaves nicely, and our limits are unique. [↑](#footnote-ref-28)